[6]Ekaterinaris, J. A., and Schiff, L. B., "Numerical Investigation of the Effects of Variation of Angle of Attack and Sweep Angle on Vortex Breakdown over Delta Wings," AIAA Paper 91-3000, 1991.

[7]Baldwin, B. S., and Lomax, H., "Thin Layer Approximation and Algebraic Model for Separated Turbulent Flows," AIAA Paper 78-257, 1978.

[8]Courant, R., Isaacson, E., and Rees, M., "On the Solution of Nonlinear Hyperbolic Differential Equations by Finite Differences," *Communications on Pure Applied Mathematics*, Vol. 5, No. 2, 1952, pp. 243–255.

[9]Chakravarthy, S. R., Anderson, D. A., and Salas, M. D., "The Split-Coefficient Matrix Method for Hyperbolic Systems of Gas Dynamic Equations," AIAA Paper 80-0286, 1980.

[10]Cunningham, A. M., and den Boer, R. G.,"Low-Speed Unsteady Aerodynamics of a Pitching Strake Wing at High Incidence—Part II: Harmonic Analysis," *Journal of Aircraft*, Vol. 27, No. 1, 1990, pp. 31–41.

Vorticity in an Inviscid Fluid at Hypersonic Speeds

David Nixon*

The Queen's University of Belfast, Belfast BT9 5A6, Northern Ireland, United Kingdom

Introduction

OVER the last several years there has been a resurgence of interest in the hypersonic aircraft and in the necessary technologies to make it a feasible proposition. One of the most critical aerodynamic unknowns is the nature of turbulence at very high speeds. There have been investigations into the effects of flow compressibility on turbulence, but most, if not all, of these are directed to extending the knowledge of turbulence at low speeds to incrementally higher speeds. Although this type of approach does yield some useful information, it does not give an indication of the possible magnitude of the problems; namely, what is turbulence in the limit as the Mach number approaches infinity? An attempt to get some indication of turbulence at high Mach numbers is reported by Childs[1] et al. This work was concerned with a numerical simulation of a free shear layer at convective Mach numbers of 4. This study indicated that turbulence at this relatively low Mach number was considerably different from low-speed turbulence. The most notable difference was that the large vortical structures that are a feature of free shear layers inclined at fairly steep angles to the dominant flow direction, in comparison with the situation at low speeds at which the structures are spanwise. However, the free shear layer may not be representative of turbulence in general because of the dominance of the shear layer by large vortical structures, a feature that is not apparent in wall-bounded flows. It is noteworthy that Nixon[2] predicted the appearance of the swept structures using an irrotational model, which may indicate that this feature is not directly connected with turbulence.

There have been other experimental studies of turbulence at high Mach numbers,[3–5] but at very high Mach numbers ($M \sim 10$–20) it is difficult to determine whether it is velocity fluctuations (the dominant factor in low-speed turbulence) or density and pressure fluctuations that are being measured.

This Note describes a simple analysis of vorticity evaluation at high Mach numbers. Since vorticity is the cornerstone of classical turbulence theories, it is helpful to establish whether these theories can be valid at very high Mach numbers. Since analysis of turbulence itself is almost impossible, a simple model problem is posed, namely, the steady flow of an inviscid fluid with constant total enthalpy (h_0) but with an arbitrary prescribed initial vorticity. It is assumed that the fluid can be accelerated isentropically to a very high Mach number where it becomes steady. Although inviscid, the model problem does relate to turbulence, since large-scale turbulence is more or less independent of viscosity.

Received Oct. 15, 1993; accepted for publication March 19, 1994.

*Department of Aeronautical Engineering.

Analysis

It is assumed in the analysis that in the region of interest the flow quantities are small perturbations from reference values, denoted by the subscript r; this will give at least a first-order effect. The density is denoted by ρ, temperature by T, specific enthalpy by h, pressure by p, and entropy by S. The gas constant is denoted by R and the total velocity by q. For a steady flow, with constant total specific enthalpy, the Euler equations can be combined with Gibbs equation to give the following relationships:

$$\frac{\rho}{\rho_r} = \left[1 + \frac{(\gamma-1)}{2} M_r^2 (1-q^2)\right]^{\frac{1}{(\gamma-1)}} \exp(-S/R) \tag{1}$$

$$\frac{T}{T_r} = \left[1 + \frac{(\gamma-1)}{2} M_r^2 (1-q^2)\right] \tag{2}$$

$$\frac{p}{p_r} = \left[1 + \frac{(\gamma-1)}{2} M_r^2 (1-q^2)\right]^{\frac{\gamma}{\gamma-1}} \exp(-S/R) \tag{3}$$

where γ is the ratio of specific heats. The velocity q is normalized with respect to a reference value q_r and has components in a Cartesian coordinate system (x, y, z) of u, v, w.

Let q' be a turbulent velocity fluctuation about q_r and normalized by q_r, that is,

$$q^2 = 1 + 2(u_r u' + v_r v' + w_r w') + u'^2 + v'^2 + w'^2 \tag{4}$$

where u_r, v_r, and w_r are the normalized velocity components of q_r.

Now let

$$\begin{aligned} \rho/\rho_r &= 1 + \rho' \\ T/T_r &= 1 + T' \\ p/p_r &= 1 + p' \\ S/R &= S_0/R + S'/R \end{aligned} \tag{5}$$

where S_0/R is the (constant) reference value of entropy. This implies that the reference flow is irrotational.

If the reasonable assumption that the general fluctuating quantity f' satisfies

$$|f'| \ll 1 \tag{6}$$

is made, then for p/p_r to be real and nonzero,

$$\frac{(\gamma-1)}{2} M_r^2 (q^2 - 1) < 1 \tag{7}$$

or

$$\begin{aligned} 0 \preceq 1 &+ u_r u' + v_r v' + w_r w' \\ &+ (u'^2 + v'^2 + w'^2)/2 < 1\big/\big[(\gamma-1)M_r^2\big] \end{aligned} \tag{8}$$

If Eq. (6) is applied, Eq. (8) becomes, to first approximation,

$$0 \preceq u_r u' + v_r v' + w_r w' < 1\big/\big[(\gamma-1)M_r^2\big] \tag{9}$$

If the flow is such that $|u_r| \gg |v_r|, |w_r|$, that is, the dominant flow is in the x direction, then Eq. (9) indicates that

$$|u_r u'| \to 0 \text{ as } M_r \to \infty \tag{10}$$

which shows that the fluctuations become two dimensional in the plane normal to the streamwise direction.

In a similar fashion, the relative magnitude of the nondimensional fluctuations in density, pressure, and temperature compared with the velocity fluctuations can be determined. Expanding Eq. (1) in a series yields

$$\rho' = S'/R - (q^2-1)M_r^2/2 + \underline{O}\left\{\left[0.5(\gamma-1)M_r^2(1-q^2)\right]^2\right\} \quad (11)$$

Then, because of Eq. (7), a first approximation to Eq. (11) can be defined as

$$\rho' = -S'/R - (q^2-1)M_r^2/2 \quad (12)$$

This assumes that the relative density is not close to zero.

From Crocco's equation, entropy is related to vorticity Ω by

$$T\nabla(S/R) = (\bar{q}x\bar{\Omega})\left(\gamma M_r^2\right) \quad (13)$$

in which q and T are the normalized velocity vector and temperature, respectively. For irrotational flow, the vorticity and the entropy gradient are zero. By assuming a rotational perturbation about an irrotational flow, in other words, the reference state is irrotational, with zero entropy gradient, one can observe that, to a first approximation,

$$\nabla x(S'/R) = (\bar{q}_r x\bar{\Omega})\left(\gamma M_r^2\right) \quad (14)$$

Note that in this approximation the normalized temperature can be assumed to be unity.

Using Eqs. (12) and (14), one can show that

$$\begin{aligned}\nabla\rho' &\sim -\nabla(S'/R) - M_r^2\nabla(q^2-1)/2\\ &= -M_r^2[\gamma\bar{q}x\bar{\Omega}' + \nabla(q^2-1)/2] \end{aligned} \quad (15)$$

Since ρ' is bounded, it can be inferred that, in the absence of shocks or contact surfaces, $\nabla p'$ is bounded; it can then be inferred from Eq. (15) that as $M_r^2 \to \infty$

$$\gamma\bar{q}x\bar{\Omega}' + \nabla(q^2-1)/2 = 0 \quad (16)$$

If q_r is dominated by u_r, that is, the dominant flow direction is in the x direction, then to first approximation Eq. (16) gives

$$\begin{aligned} u_r\frac{\partial u'}{\partial x} &= 0\\ u_r\frac{\partial u'}{\partial y} &= \gamma u_r\Omega_3'\\ u_r\frac{\partial u'}{\partial z} &= -\gamma u_r\Omega_2' \end{aligned} \quad (17)$$

where Ω_1, Ω_2, and Ω_3 are in the x, y, and z directions, respectively.

Consider now the vorticity transport equation

$$\begin{aligned} L(\bar{\Omega}) &= \bar{\Omega}_t + (\bar{q}.\nabla)\bar{\Omega} - (\bar{\Omega}\cdot\nabla)\bar{q} + \bar{\Omega}(\nabla\cdot\bar{q})\\ &= -\nabla x(\nabla p/\rho)/\gamma M_r^2 \end{aligned} \quad (18)$$

The term on the right can be evaluated by combining Eqs. (1) and (3) to give

$$-\nabla x(\nabla p/\rho)/\gamma M_r^2 = (\gamma-1)\nabla(S/R)x\nabla q^2/2 \quad (19)$$

or, by using Eq. (13),

$$-\nabla x(\nabla p/\rho)/\gamma M_r^2 = (\gamma-1)\left(\gamma M_r^2/T\right)(\bar{q}x\bar{\Omega})x\nabla q^2/2 \quad (20)$$

Using Eq. (20) in Eq. (18) and assuming that the normalized vorticity and velocity are bounded, then one may see that as $M_r^2 \to \infty$

$$(\bar{q}x\bar{\Omega})x\nabla q^2 \to 0 \quad (21)$$

If $\bar{q}$ is dominated by u_r, then Eq. (21) gives the following possibilities:

$$\bar{\Omega} = 0 \quad (22a)$$

$$u_r\Omega_2 = u_r\Omega_3 = 0 \quad (22b)$$

$$u_r\Omega_2\frac{\partial q^2}{\partial y} - u_r\Omega_3\frac{\partial q^2}{\partial z} = 0 \quad (22c)$$

Equation (22a) states there is zero vorticity, and Eq. (22b) indicates that streamwise vortices dominate, that is, a Beltrami flow exists. If the problem is a geometrically two-dimensional flow, such that

$$\left|\frac{\partial q^2}{\partial z}\right| \ll \left|\frac{\partial q^2}{\partial y}\right| \quad (23)$$

then Eq. (22c) indicates that

$$u_r\Omega_2\frac{\partial q^2}{\partial y} = 0 \quad (24)$$

or

$$\Omega_2 = 0 \quad (25)$$

This indicates that vortices may either be streamwise or spanwise. If Eqs. (10) and (17) are combined, it may be seen that

$$\Omega_2 = \Omega_3 = 0 \quad (26)$$

again leaving the only possibility that the vortices are predominantly in the streamwise direction.

Physically, the analysis indicates that once vorticity is introduced into the flow, its distribution is dominated by the entropy relation that restricts the type of vorticity allowed. To a first approximation, as the Mach number tends to infinity in a typical two-dimensional shear flow, the vortices are aligned with the mean streamwise direction, a direction that causes no entropy production.

Concluding Remarks

A simple analysis has shown that for steady inviscid flow there is only one dominant vorticity component as the Mach number approaches infinity. This is in stark contrast to low-speed flows where vorticity in all three directions can be of equal magnitude and in fact interchange energy. Since almost all of the classic explanations of turbulence are based on vortex dynamics, the present work indicates that such an approach may not be valid at high speeds.

Acknowledgments

Part of this work was performed under Air Force Office of Scientific Research Contract F 49620-91-C-0037. This work was performed while the author was at Nielsen Engineering and Research Inc., Mountain View, California.

References

[1]Childs, R. E., Nixon, D., Keefe, L. R., and Rodman, L. C., "A Study of Compressible Turbulence," AIAA Paper 93-0659, 1993.

[2]Nixon, D.,"A Theory for the Mixing of a Compressible Round Jet," AIAA Paper 94-2193, 1994.

[3]Morkovin, M. V., "The Effects of Compressibility on Turbulent Flows," *The Mechanics of Turbulence*, edited by A. Foree, Gordon & Breach, New York, 1964.

[4]Harvey, J., Bergman, R. C., and Holden, M. S., "An Experimental Study of Hypersonic Turbulence on a Sharp Cone," AIAA Paper 89-1866, 1989.

[5]Laderman, A. J., and Demetriades, A., "Mean and Fluctuating Flow Measurements in the Hypersonic Boundary Layer over a Cooled Wall," *Journal of Fluid Mechanics*, Vol. 63, Pt. 1, 1974.